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VARIATIONAL PRINCIPLES OF THE NONLINEAR THEORY OF ELASTICITY. CASE OF SUPERPOSITION OF A SMALL DEFORMATION ON A FINITE DEFORMATION

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General relationships and variational theorems in the theory of small elastic deformations of an elastic solid, superposed on a finite deformation, are pres - ented. A relationship is established between the two modes of the equilibrium equation: in the metric of the undeformed state, and in the metric of an initially deformed state of the solid. A formula is obtained from the potential energy accumulated in an elastic prestressed solid for a small deformation. Variational principles, analogous to the variational principles of the theory of finite deformations [1] and differing from the variational theorems of classical theory of elasticity by the nonsymmetry of the dual tensors, are formulated.

A generalization of the Clapeyron and Betti theorem to the case of small deformations of a prestressed elastic solid is obtained. The formulated variational principles refer, in particular, to the problem of bifurcation of equilibrium of a nonlinearly elastic solid.

Let v be a volume occupied by an elastic solid in an undeformed state, and V° the volume it occupies after deformation caused by mass forces $\mathbf{K}^\circ$ and surface forces $\mathbf{F}^\circ$ ($\mathbf{F}^\circ$ is the vector of forces per unit area of the undeformed solid).

An equilibrium state given by the radius vector of a point of the deformed solid

$$\mathbf{R} = \mathbf{R}^0 + \eta \mathbf{w}$$

adjacent to the equilibrium state of the V°- volume is considered, where η is a small parameter.

Two modes of the equations describing the small deformation of an elastic solid in the presence of initial stresses are given in [3 - 5]. In the first case the equations are written in the metric of the state of the v-volume

$$\nabla\cdot \mathrm{D}^{\cdot} + \rho_0 \mathbf{k} = 0 \quad \text{в } v, \qquad \mathbf{n}\cdot \mathrm{D}^{\cdot} = \mathbf{f} \quad \text{on } o$$
$$\mathrm{D}^{\cdot} = \left\{\frac{d}{d\eta}\,[\mathrm{D}(\mathbf{R}^0 + \eta\mathbf{w})]\right\}_{\eta=0} \tag{1}$$

Here ∇ is the nabla operator in the metric of the v- volume, $\mathbf{w}$ is the vector of the additional displacement, D the Piola stress tensor [4], ρ_0 the density of the material in the undeformed state, o the surface bounding the volume v, $\mathbf{n}$ the normal to the surface o, $\mathbf{k}$ the additional mass force ($\mathbf{K} = \mathbf{K}^\circ + \eta\mathbf{k}$), $\mathbf{f}$ the additional surface force computed per unit area of the surface o ($\mathbf{F} = \mathbf{F}^\circ + \eta\mathbf{f}$). In the second case the equations are written in the metric of the V°-volume

$$\nabla'\cdot\Theta + \rho\mathbf{k} = 0 \quad \text{в } V^\circ, \qquad \mathbf{N}\cdot\Theta = \mathbf{f}' \quad \text{on } O$$
$$\Theta = \mathrm{T}^{\cdot} + \nabla'\cdot\mathbf{w}\mathrm{T}^\circ - \nabla'\mathbf{w}^T\cdot\mathrm{T}^\circ \tag{2}$$

Here O is the surface bounding the volume V°, ∇' is the nabla operator in the metric of the V°-volume, $\mathbf{N}$ the normal to the surface O, ρ is the density of the material in the state of the V°-volume, $\mathbf{f}'$ the additional surface force per unit surface O ($\mathbf{F}' = \mathbf{F}'^\circ + \eta\mathbf{f}'$) and T is the Cauchy stress tensor.

Let us establish the connection between the tensors $\mathrm{D}^{\cdot}$ and Θ on the basis of the relationship [4]

$$\mathrm{D} = \sqrt{G/g}\,(\nabla\mathbf{R}^T)^{-1}\cdot\mathrm{T} \tag{3}$$

Here $\sqrt{G/g}$ is the ratio between the volume elements of the deformed and undeformed solid. Let us differentiate (3) with respect to the parameter η

$$\mathrm{D}^{\cdot} = (\sqrt{G/g})^{\cdot}\,(\nabla\mathbf{R}^T)^{-1}\cdot\mathrm{T} + \sqrt{G/g}\,(\nabla\mathbf{R}^T)^{-1}\cdot\mathrm{T}^{\cdot} -$$
$$- \sqrt{G/g}\,(\nabla\mathbf{R}^T)^{-1}\cdot\nabla\mathbf{w}^T\,(\nabla\mathbf{R}^T)^{-1}\cdot\mathrm{T}$$

Taking into account the connection between the nabla operators

$$\nabla = (\nabla\mathbf{R})\cdot\nabla' \tag{4}$$

and the relationship [5]

$$(\sqrt{G/g})^{\cdot} = \sqrt{G/g}\,\nabla'\cdot\mathbf{w}$$

we obtain

$$\mathrm{D}^{\cdot} = \sqrt{G^\circ/g}\,(\nabla\mathbf{R}^{\circ T})^{-1}\cdot(\mathrm{T}^{\cdot} + \nabla'\cdot\mathbf{w}\mathrm{T}^\circ - \nabla'\mathbf{w}^T\cdot\mathrm{T}^\circ) = \sqrt{G^\circ/g}\,(\nabla\mathbf{R}^{\circ T})^{-1}\cdot\Theta$$

Therefore the tensors $\mathrm{D}^{\cdot}$ and Θ are connected by the same relationship as the tensors D° and T°. In an ideally elastic solid the Piola stress tensor is connected to the tensor-gradient of the radius-vector of a point of the deformed solid as follows:

$$\mathrm{D} = dW/d\mathrm{C}, \qquad \mathrm{C} = \nabla\mathbf{R} \tag{6}$$

Here W is the specific potential energy per unit volume prior to deformation. Let $\mathbf{r}_s$ be the vector basis of the v-volume

$$\mathrm{D} = \partial^{st}\, \mathbf{r}_s \mathbf{r}_t, \qquad \mathrm{C} = \mathrm{C}_{st}\, \mathbf{r}^s \mathbf{r}^t$$

According to (6) we have

$$\partial^{\cdot st} = \frac{\partial^2 W}{\partial C_{st} \partial C_{mn}}\, \dot{C}_{mn}, \qquad \dot{C}_{mn} = \nabla_m w_n$$

Here ∇_m is the symbol of the covariant derivative in the metric of the v-volume. A tensor of fourth rank with the components

$$K^{st\,mn} = \partial^2 W / \partial C_{st} \partial C_{mn}$$

will evidently possess the following symmetry property:

$$K^{st\,mn} = K^{mn\,st} \tag{7}$$

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two different vectors. Let us use the notation

$$\mathrm{D}^{\cdot}(\mathbf{w}_i) = K^{st\,mn} \nabla_m w_{in} \mathbf{r}_s \mathbf{r}_t \qquad (i = 1,2)$$

and let us prove the following reciprocity relationship

$$\mathrm{D}^{\cdot}(\mathbf{w}_1) \cdot\cdot \nabla \mathbf{w}_2^T = \mathrm{D}^{\cdot}(\mathbf{w}_2) \cdot\cdot \nabla \mathbf{w}_1^T \tag{8}$$

In fact, utilizing (7), we have

$$\mathrm{D}^{\cdot}(\mathbf{w}_1) \cdot\cdot \nabla \mathbf{w}_2^T = K^{st\,mn} \nabla_m w_{1n} \nabla_s w_{2t} =$$
$$= K^{mn\,st} \nabla_s w_{2t} \nabla_m w_{1n} = K^{st\,mn} \nabla_m w_{2n} \nabla_s w_{1t} = \mathrm{D}^{\cdot}(\mathbf{w}_2) \cdot\cdot \nabla \mathbf{w}_1^T$$

From (4), (5) and (8) it is easy to obtain the reciprocity relationship for the tensor

$$\Theta(\mathbf{w}_1) \cdot\cdot \nabla' \mathbf{w}_2^T = \Theta(\mathbf{w}_2) \cdot\cdot \nabla' \mathbf{w}_1^T \tag{9}$$

The potential energy of an elastic solid for dead external forces is written as

$$\Pi = \iiint_v W\, d\tau_0 - \iiint_v \rho_0 \mathbf{K} \cdot \mathbf{u}\, d\tau_0 - \iint_o \mathbf{F}^\circ \cdot \mathbf{u}\, do$$

where $\mathbf{u}$ is the displacement vector. Let us evaluate the increment in the potential energy upon communicating an additional displacement $\eta \mathbf{w}$ to the points to second order accuracy:

$$\Pi - \Pi(\mathbf{R}^\circ) = \Pi_1 \eta + \Pi_2 \eta^2 + \ldots, \quad \Pi_1 = \Pi^{\cdot}(\mathbf{R}^\circ), \quad \Pi_2 = {}^1/_2\ \Pi^{\cdot\cdot}(\mathbf{R}^\circ)$$

Referring to (6) we have

$$\Pi^{\cdot} = \iiint_v \mathrm{D} \cdot\cdot \nabla \mathbf{w}^T d\tau_0 - \iiint_v \rho_0 \mathbf{K} \cdot \mathbf{w}\, d\tau_0 - \iint_o \mathbf{F}^\circ \cdot \mathbf{w}\, do$$

$$\Pi^{\cdot\cdot} = \iiint_v \mathrm{D}^{\cdot} \cdot\cdot \nabla \mathbf{w}^T d\tau_0 \qquad (\mathbf{K}^{\cdot} = \mathbf{F}^{\cdot} = 0)$$

Furthermore, integrating by parts, we obtain

$$\Pi_1 = \iint_o (\mathbf{n} \cdot \mathrm{D}^\circ - \mathbf{F}^\circ) \cdot \mathbf{w}\, do - \iiint_v (\nabla \cdot \mathrm{D}^\circ + \rho_0 \mathbf{K}^\circ) \cdot \mathbf{w}\, d\tau_0 = 0 \tag{10}$$

since the state prescribed by the vector $\mathbf{R}^\circ$ is in equilibrium. Therefore, the following expression holds for the potential energy accumulated in a prestressed elastic solid under small deformation and in the absence of additional external forces

$$\Pi_2 = \frac{1}{2} \iiint_V \mathrm{D}^{\cdot} \cdot\cdot \nabla \mathbf{w}^T \, d\tau_0 = \frac{1}{2} \iiint_{V^\circ} \Theta \cdot\cdot \nabla' \mathbf{w}^T \, d\tau \tag{11}$$

The last equality in (11) follows from (4), (5). Since the tensor $\mathrm{D}^{\cdot}$ depends linearly on the tensor $\nabla\mathbf{w}$, then the quantities

$$W_2 = {}^1/_2\, \mathrm{D}^{\cdot} \cdot\cdot \nabla\mathbf{w}^T, \qquad W_2' = {}^1/_2\, \Theta \cdot\cdot \nabla'\mathbf{w}^T \tag{12}$$

are quadratic forms of the tensor components $\nabla\mathbf{w}$ and $\nabla'\mathbf{w}$ respectively. Keeping in mind the reciprocity relationship (8), let us evaluate the variation W_2

$$\delta W_2 = {}^1/_2\, \mathrm{D}^{\cdot}(\mathbf{w}) \cdot\cdot \nabla\delta\mathbf{w}^T + {}^1/_2\, \mathrm{D}^{\cdot}(\delta\mathbf{w}) \cdot\cdot \nabla\mathbf{w}^T = \mathrm{D}^{\cdot}(\mathbf{w}) \cdot\cdot \delta\nabla\mathbf{w}^T$$

It hence follows that the tensor $\mathrm{D}^{\cdot}$ is a potential tensor function of the tensor $\nabla\mathbf{w}$

$$\mathrm{D}^{\cdot} = dW_2 / d\mathrm{L}, \qquad \mathrm{L} = \nabla\mathbf{w} \tag{13}$$

Analogously, we arrive at the equality

$$\Theta = dW_2' / d\mathrm{L}', \qquad \mathrm{L}' = \nabla'\mathbf{w} \tag{14}$$

Let us assume that the initial stresses in the solid are such that the tensors L and L', respectively, can be expressed in terms of the tensors $\mathrm{D}^{\cdot}$ and Θ from (13), (14). Let us introduce the functions of the tensors $\mathrm{D}^{\cdot}$ and Θ

$$B_2(\mathrm{D}^{\cdot}) = \mathrm{D}^{\cdot} \cdot\cdot \nabla\mathbf{w}^T - W_2, \qquad B_2'(\Theta) = \Theta \cdot\cdot \nabla'\mathbf{w}^T - W_2' \tag{15}$$

Then according to the property of the Legendre transformation, we obtain

$$\mathrm{L} = dB_2 / d\mathrm{D}^{\cdot}, \qquad \mathrm{L}' = dB_2' / d\Theta \tag{16}$$

The presence of the relationships (13), (14) and (16) permits formulation of variational principles of the theory of small deformations of a prestressed elastic solid.

These variational principles are completely analogous to those contained in [1] referring to the theory of finite elastic deformations; it is sufficient to perform the following change of notation in [1]: $\nabla\mathbf{R}$ replaced by $\nabla\mathbf{w}$ (or $\nabla'\mathbf{w}$), $\mathbf{u}$ by $\mathbf{w}$, D by $\mathrm{D}^{\cdot}$ (or Θ), C by L (or L'), W by W_2(or W_2'), B by B_2 (or B_2'), $\mathbf{F}^\circ$ by $\mathbf{f}$ (or $\mathbf{f}'$), $\mathbf{K}$ by $\mathbf{k}$. The notation in parentheses refers to writing the functionals in the metric of the V°-volume). If there are no additional external forces, then the variational theorems presented correspond to the problem of bifurcation of the equilibrium of a nonlinearly elastic solid. For a particular case (semilinear material under affine initial deformation) the second variational principle in the theory of small deformations with initial stresses was formulated in [6]. The appropriate explicit expression for the functional J_2 is presented there.

Let us note still certain relationships which generalize known theorems of classical linear elasticity theory to the case of the theory of small deformations imposed on a finite deformation. An identity analogous to the Clapeyron theorem

$$\iiint_{V^\circ} W_2'\, d\tau = \frac{1}{2} \iiint_{V^\circ} \Theta \cdot\cdot \nabla'\mathbf{w}^T \, d\tau = \frac{1}{2} \iiint_{V^\circ} \rho\mathbf{k} \cdot \mathbf{w}\, d\tau + \frac{1}{2} \iint_O \mathbf{f}' \cdot \mathbf{w}\, dO \tag{17}$$

follows from (2), (12) and the integration by parts formula. Let points of an elastic solid receive a small additional displacement $\mathbf{w}_1$ under the effect of the additional forces $\mathbf{k}_1$ and $\mathbf{f}_1'$, and the displacement $\mathbf{w}_2$ under the effect of the system of forces $\mathbf{k}_2$, $\mathbf{f}_2'$. A theorem on reciprocity of work of the additional forces on the additional displacements follows from (2), (9)

$$\iiint\limits_{V^\circ} \rho \mathbf{k}_1 \cdot \mathbf{w}_2 \, d\tau + \iint\limits_{O} \mathbf{f}_1' \cdot \mathbf{w}_2 \, dO = \iiint\limits_{V^\circ} \rho \mathbf{k}_2 \cdot \mathbf{w}_1 \, d\tau + \iint\limits_{O} \mathbf{f}_2' \cdot \mathbf{w}_1 \, dO \qquad (18)$$

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ON THE THEORY OF ELASTICITY OF INHOMOGENEOUS MEDIA

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The general solution is obtained of the equilibrium equations in displacements for inhomogeneous isotropic media, whose elastic characteristics are differentiable functions of the Cartesian coordinates. It is shown that the components of the displacement vector in the three-dimensional problem of elasticity theory can always be expressed in terms of two functions which satisfy second and fourth order linear partial differential equations, respectively.

Of the earlier research devoted to analogous problems, the paper [1] should first be noted in which an equation is derived for the Airy stress function in the two-dimensional problem of the theory of elasticity of an inhomogeneous medium. A general solution of the equilibrium equations in displacements is obtained in [2, 3] for the case of axisymmetric deformation of bodies of revolution whose elastic moduli vary exponentially as a function of the coordinate z. A power-law change in the elastic modulus was investigated in [4] with primary attention